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International Journal of Heat and Mass Transfer 49 (2006) 1022–1026

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Resonance properties of soluble gas bubbles

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Received 25 April 2005; received in revised form 16 August 2005
Available online 15 November 2005

Abstract

Soluble gas bubbles in a liquid experiencing radial oscillations created by an acoustic field are considered. It is shown that the resonance frequency of large soluble gas bubbles practically coincides with the natural frequency of gas bubbles as determined by the Minnaert formula. In the case of small gas bubbles, the presence of capillary effects and solubility of the gas in the liquid leads to a new resonance frequency that differs from the Minnaert frequency. A simple analytic formula is obtained that relates the resonance frequency of a soluble gas bubble and its radius.

Keywords: Resonance; Soluble gas bubble; Diffusion

1. Introduction

The study of oscillations of gas bubbles in a liquid is of considerable practical interest specifically in regard to the problem of the use of bubble screens for damping shock waves and the use of acoustic disturbances for intensification of technological processes.

It is of particular significance for the polishing of small metallic manufactured wares when removing agnails from their surface [1].

The bubbles are also widely used in bubble chambers for the registration of tracks of charged particles. This is based on the phenomenon that charged particles passing through a cryogenic liquid create invisible microbubbles along their trajectory. Under the influence of an acoustic field these bubbles are oscillating. At the same time, their mean radii slowly increase. Thus, they become visible after a large number of radial pulsations [2,3].

The behavior of soluble gas bubbles in a liquid has been examined in a number of studies (see reviews [4,5]). The influence of gas solubility on the resonance properties of the soluble gas bubbles making small radial oscillations under the influence of an acoustic field is investigated in this paper.

Resonant properties of vapor bubbles were studied in a number of works (see for example [6,7] where the existence of two resonances was established on the basis of numerical calculations). For the first time analytic formula, revealing correlation between the second resonance frequency of a vapor bubble and its radius was derived in [8]. Later a formula of the same type was published in [9].

2. Basic equations

The problem of spherically symmetric processes around gas–vapor bubbles has been formulated in [10–12].

We shall take the assumptions made in the Rayleigh formulation for the dynamics of a single bubble [13].

Conservation of the spherical symmetry of the process; uniformity of the pressure within the bubble (homobaricity). In the course of bubble oscillations, homobaricity prevails when the size of the bubble is much less than the length of a sound wave in the gas [10,11].

At the same time it is supposed according to the equation of state that the gas density at each point corresponds to its temperature.

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doi:10.1016/j.ijheatmasstransfer.2005.09.008

Nomenclature

R	bubble radius	γ	specific heat ratio
$\dot{R}$	time derivative of the radius	c	specific heat
r	radial Euler coordinate	C_p	specific heat of the gas at constant pressure
t	time	f	frequency of oscillations
T	temperature	ω	circular frequency
ρ	density	P_A	acoustic pressure amplitude
p	pressure	α	non-dimensional displacement of bubble surface
k	concentration of the gas in the liquid	σ	surface tension coefficient
D	diffusion coefficient	S	resonance function
Γ	Henry's coefficient		
w	radial velocity	*Subscripts*	
v	kinematic viscosity	l	liquid
λ	thermal conductivity	g	gas
a	thermal diffusivity	s	at saturation
B	gas constant	σ	on bubble surface
j	rate of mass transfer per unit interface surface	O	at equilibrium
μ	viscosity	∞	conditions at infinity

This statement of the problem when the pressure uniformity and the temperature and gas density non-uniformities in the bubble is assumed, is valid for a wide range of bubble sizes. It has been estimated [14] that the characteristic time of temperature equalization in the bubble considerably exceeds the time of pressure equalization.

Within the framework of these assumptions the diffusion equation for the gas in the liquid in the spherical Euler coordinates (r, t) takes the form:

$$\frac{\partial k}{\partial t} + w_e \frac{\partial k}{\partial r} = \frac{D}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial k}{\partial r}\right), \quad r > R \tag{2.1}$$

The boundary conditions at the surface of the bubble and at infinity are as follows:

$$r = R(t) : k = k_s = \Gamma p, \quad r = \infty : k = k_O = \Gamma p_O \tag{2.2}$$

The velocity of the bubble surface and the mass velocity of the gas and the liquid at that surface w are related by the expressions [10]:

$$\dot{R} = w_{g\sigma} - \frac{j}{\rho(R)}, \quad j = -D\frac{\partial k}{\partial r}(R),$$
$$\dot{R} = w_{e\sigma} - j/\rho_e, \quad w_e = w_{e\sigma} R^2/r^2 \tag{2.3}$$

The heat equation and the equation of state of the gas take the form:

$$\rho C_p\left(\frac{\partial T}{\partial t} + w\frac{\partial T}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda r^2 \frac{\partial T}{\partial r}\right) + \frac{\mathrm{d}p}{\mathrm{d}t}$$
$$p(t) = \rho(r,t) B T(r,t) \quad O < r < R \tag{2.4}$$

For a gas bubble without phase transitions it is legitimate to make the simplifying assumption [14,15] $T_\sigma = T_O$. This makes it possible to get by with solving only the interior thermal problem (in the gas). This is connected with the fact that the thermal conductivity of the liquid is much greater than that of the gas, while the thermal diffusivity of the liquid is much less than that of the gas [15].

Accordingly the boundary conditions for the heat Eq. (2.4) can be written in the form:

$$r = R(t) : T = T_O, \quad r = O : \frac{\partial T}{\partial r} = O \tag{2.5}$$

The equations for the pressure and the velocity profile of the gas in the bubble take the form [10,11]:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{3(\gamma-1)}{R}\left(\lambda\frac{\partial T}{\partial r}\right)_\sigma - \frac{3\gamma p}{R} w_{g\sigma}$$
$$w(r,t) = \frac{r}{R} w_{g\sigma} + \frac{\gamma - 1}{\gamma}\left[\lambda\frac{\partial T}{\partial r} - \frac{r}{R}\left(\lambda\frac{\partial T}{\partial R}\right)_\sigma\right] \tag{2.6}$$

The equation of radial pulsations of the bubble in an incompressible liquid takes the form:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{p - P(\infty) - 2\sigma/R}{\rho_e} - \frac{4\mu}{\rho_e R}\dot{R} \tag{2.7}$$

3. Oscillations of soluble gas bubble in an acoustic field

The diffusion is very slow process and can manifest itself at very low frequencies and for very small bubbles only. For this reason we can simplify the system (2.1)–(2.7) assuming that the process is isothermal.

For uniform isothermal soluble gas bubble the system of equations will include Eqs. (2.1), (2.2) and (2.7) and the following equations (equations of state and conservation of mass):

$$p_g = \rho_g B T_O \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi R^3 \rho_g\right) = -4\pi R^2 j, \quad j = -D\frac{\partial k}{\partial r}(R) \tag{3.2}$$

We assume the pressure amplitude of the acoustic field P_A is small by comparison with the static pressure in the liquid P_∞:

$$P(\infty) = P_\infty + P_A e^{i\omega t}, \quad P_A \ll P_\infty \tag{3.3}$$

In this case, the radius of the bubble may be described by the real part of the expression

$$R = R_O(1 + \alpha e^{i\omega t}), \quad |\alpha| \ll 1 \tag{3.4}$$

where α is a complex number.

Within the framework of a linear representation, analytic solution of the basic system of equations can be obtained, similarly as done in [7] for the vapor bubble.

Let P and K be small deviations of the pressure, and gas concentration in the liquid from the equilibrium state

$$p = p_0[1 + Pe^{i\omega t}], \quad k = k_0[1 + K(r)e^{i\omega t}] \tag{3.5}$$

We assume that

$$j = \bar{j}e^{i\omega t} \tag{3.6}$$

where

$$p_0 = P_\infty + \frac{2\sigma}{R_0}$$

Let us linearize the system of basic equations taking into account relations (3.3)–(3.6). After transformation to non-dimensional variables the diffusion equation can be rewritten as follows:

$$\frac{\partial K}{\partial \tau} = \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial K}{\partial \xi}\right), \quad \xi > 1$$
$$\frac{dK_s}{dP} = 1, \quad \xi = \frac{r}{R}, \quad \tau = \frac{t\omega R_0^2}{D} \tag{3.7}$$

Taking into account (3.5) and (3.6) we can present the diffusion equation in form:

$$AK = \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{dK}{d\xi}\right), \quad A = \frac{i\omega R_0^2}{D} \tag{3.8}$$

Solution of this equation has a form:

$$K = K_s \frac{\exp[A(1-\xi)]}{\xi} \tag{3.9}$$

Calculating the derivative of gas concentration at the bubble surface we will obtain

$$\left.\frac{dK}{d\xi}\right|_1 = K_s(-\sqrt{A} - 1) \tag{3.10}$$

The mass flux will have a form:

$$j = -D\frac{k_0}{R_0}\left.\frac{dK}{d\xi}\right|_1 = -\frac{Dk_0}{R_0}K_s(-\sqrt{A} - 1) \tag{3.11}$$

From (3.7) we will receive $K_s = P$. So,

$$j = -D\frac{k_0}{R_0}P(-\sqrt{A} - 1) \tag{3.12}$$

Another equations of the system of basic equations will have a form

$$\dot{R}\rho_g + \frac{R}{3}\dot{\rho}_g = -j \tag{3.13}$$

$$R_0\ddot{R} = \frac{p_0(1 + Pe^{i\omega t}) - P_\infty - P_A e^{i\omega t} - \dfrac{2\sigma}{R_0(1 + \alpha e^{i\omega t})}}{\rho_e} - 4\frac{\nu}{R_0}\dot{R} \tag{3.14}$$

$$\frac{p_g}{p_0} = \frac{\rho_g}{\rho_{g0}} \tag{3.15}$$

Eqs. (3.13)–(3.15) can be transformed to the form:

$$\dot{\rho}_g = \rho_{g0}Pi\omega e^{i\omega t} \tag{3.16}$$

$$R_0\alpha i\omega\rho_{g0}e^{i\omega t} + \frac{R_0}{3}\rho_{g0}Pi\omega e^{i\omega t} = j \tag{3.17}$$

$$-R_0^2\omega^2\alpha = \frac{p_0}{\rho_e}P - \frac{P_A}{\rho_e} + \frac{2\sigma}{R_0\rho_e}\alpha - 4\nu i\omega\alpha \tag{3.18}$$

After substitution of (3.12)–(3.17) we will obtain the system of two linear algebraic equations relative α and P:

$$R_0\alpha i\omega\rho_{g0} + \frac{R_0}{3}\rho_{g0}Pi\omega = \frac{Dk_0}{R_0}P(-\sqrt{A} - 1) \tag{3.19}$$

$$-R_0^2\omega^2\alpha = \frac{p_0}{\rho_e}P - \frac{P_A}{\rho_e} + \frac{2\sigma}{R_0\rho_e}\alpha - 4\nu i\omega\alpha \tag{3.20}$$

From (3.19) we will obtain

$$P = -\frac{R_0 i\omega\rho_{g0}\alpha}{\dfrac{R_0}{3}\rho_{g0}i\omega + \dfrac{Dk_0}{R_0}(\sqrt{A} + 1)} \tag{3.21}$$

Finally, formula for the amplitude of oscillations will have a form:

$$\alpha = \frac{P_A}{-\dfrac{p_0 R_0 i\omega\rho_{g0}}{\dfrac{R_0}{3}\rho_{g0}i\omega + \dfrac{Dk_0}{R_0}(\sqrt{A} + 1)} + \rho_e R_0^2\omega^2 + \dfrac{2\sigma}{R_0} - 4i\omega\mu} = \frac{P_A}{S} \tag{3.22}$$

From (3.22) we can see that

$$\lim_{R_0 \to 0}|\alpha| = \lim_{R_0 \to \infty}|\alpha| = 0 \tag{3.23}$$

From (3.23), it follows that if $\sigma \neq 0$, there exists at least one bubble dimension at any finite frequency, such that $|\alpha|$ attains its maximal value.

Let's simplify formula (3.22). First, we will consider the case of large bubbles, in which we have the following condition fulfilled:

$$\left|\frac{R_0}{3}\rho_{g0}i\omega\right| \gg \left|\frac{Dk_0}{R_0}(\sqrt{A} + 1)\right| \tag{3.24}$$

Then

$$S = \rho_e\omega^2 R_0^2 - 3p_0 + \frac{2\sigma}{R_0} - 4i\omega\mu \tag{3.25}$$

and resonant frequency will be close to Minnaert isothermal frequency

$$\omega_1 = \frac{1}{R_0}\sqrt{\frac{3p_0}{\rho_e}} \tag{3.26}$$

Next, we will consider the other limiting case of sufficiently small bubbles, in which we have the following conditions fulfilled:

$$\left|\frac{R_0}{3}\rho_{g0}\mathrm{i}\omega\right| \ll \left|\frac{Dk_0}{R_0}\sqrt{A}\right|,\quad |A| \gg 1,$$
$$4w\mu \ll \frac{2\sigma}{R_0},\quad \rho_e R_0^2\omega^2 \ll \frac{2\sigma}{R_0} \tag{3.27}$$

Then, from (3.22) we will obtain

$$S = -\frac{R_0^2\mathrm{i}\omega\rho_{g0}}{Dk_0\sqrt{A}} + \frac{2\sigma}{R_0 p_0} \tag{3.28}$$

The resonance frequency of a vapor bubble has previously [6] been found by solving the equation $\mathrm{Re}(S) = 0$.

Such an approach for determining the resonance frequency is incorrect. It is clear from the fact that in addition to the real part $\mathrm{Re}(S)$, the imaginary part of the resonance function is also a function of the bubble radius and frequency of the acoustic field. However, this approach does not lead to major errors only in the region of large bubble radii, at which the Minnaert formula holds true. The same inaccuracy in a different article [16] led to an incorrect formula that related the resonance frequency of the bubble to its radius.

The resonance frequency should be found by solving the equation

$$\partial|S|/\partial\omega = 0 \tag{3.29}$$

and verifying of the condition

$$\partial^2|S/\partial\omega^2 > 0 \tag{3.30}$$

Let's transform the formula (3.28)

$$\alpha = \frac{P_A/p_0}{\frac{-R_0^2\mathrm{i}\omega\rho_{g0}}{Dk_0R_0\sqrt{\frac{\omega}{D}\frac{(1+\mathrm{i})}{\sqrt{2}}}} + \frac{2\sigma}{R_0p_0}} = \frac{P_A/p_0}{\frac{-\mathrm{i}(1-\mathrm{i})R_0\omega\rho_{g0}}{Dk_0\sqrt{\frac{\omega}{D}}\sqrt{2}} + \frac{2\sigma}{R_0p_0}} \tag{3.31}$$

$$S = \frac{-(1+\mathrm{i})R_0\sqrt{\omega}\rho_{g0}}{\sqrt{Dk_0}\sqrt{2}} + \frac{2\sigma}{R_0p_0} \tag{3.32}$$

$$|S|^2 = \left(\frac{R_0\rho_{g0}}{k_0}\sqrt{\frac{\omega}{2D}}\right)^2 + \left(\frac{2\sigma}{R_0p_0} - \frac{R_0\rho_{g0}}{k_0}\sqrt{\frac{\omega}{2D}}\right)^2 \tag{3.33}$$

$$= \frac{R_0^2\omega\rho_{g0}^2}{Dk_0^2} + \left(\frac{2\sigma}{R_0p_0}\right)^2 - 2\frac{2\sigma}{p_0}\frac{\rho_{g0}}{k_0}\sqrt{\frac{\omega}{2D}} \tag{3.34}$$

$$2|S|\frac{\partial|S|}{\partial\omega} = \frac{R_0^2\rho_{g0}^2}{Dk_0^2} - \frac{\sqrt{2}\sigma\rho_{g0}}{p_0\sqrt{\omega D}k_0} = 0 \tag{3.35}$$

Taking into account that $k_0 = \Gamma_{P_0}$, we will obtain from Eq. (3.35)

$$\omega_2 = \frac{2D}{R_0^4}\left(\frac{\sigma\Gamma}{\rho_{g0}}\right)^2 \tag{3.36}$$

Condition (3.30) for the minimum of resonance function (maximum of the amplitude) is satisfied.

Similarly we may obtain a formula for the resonance dimension of soluble gas bubbles oscillating in sufficiently low frequencies of an acoustic field.

For this purpose, we solve the equation

$$\partial|S|/\partial R_0 = 0 \tag{3.37}$$

Solving this equation for the function (3.34) we will obtain

$$2|S|\frac{\partial|S|}{\partial R_0} = \frac{2R_0\omega\rho_{g0}^2}{Dk_0^2} + \frac{4\sigma^2}{p_0^2}(-2R_0^{-3}) = 0 \tag{3.38}$$

From (3.38) we will receive

$$R_0^4 = \frac{4D}{\omega}\left(\frac{\sigma\Gamma}{\rho_{g0}}\right)^2 \tag{3.39}$$

The resulting dependence is not exactly the reverse of (3.36), as it differs by a numerical coefficient.

Substituting (3.36) in (3.31) we may determine the resonance value of the oscillations amplitude $|\alpha|P_\infty/P_A$

$$M_\omega = |\alpha|\frac{P_\infty}{P_A} = \frac{P_\infty R_0}{\sqrt{2}\sigma} \tag{3.40}$$

Let's compare the first (3.26) and (3.36) second resonant frequencies for an air bubble in water with $R_0 = 10^{-3}$ mm. The comparison shows that ω_1 is of order 10^4 kHz, but ω_2 is of order 1 Hz only.

There is an analogy between the bubble oscillating in the acoustic field and the mass oscillating on the spring. The role of the mass plays the added mass of the liquid. The role of the spring's stiffness plays elasticity of the gas inside the bubble.

In the case of soluble gases their elasticity depends on diffusion. Since diffusion is a very slow process it can manifest itself at very low frequencies only. At high frequencies the behavior of a soluble gas bubble is the same as in the case of bubble with constant mass. In this case the Minnaert resonant frequency only takes place.

It should be also noted that the second resonance has a low Q factor. Moreover, under these conditions soluble gas bubbles can be unstable. It is possible to show that the rate of growth of the amplitude of the bubble oscillations as a result of instability is of the same order as the second resonance frequency.

Consequently, the experimental verification of the second resonance is problematic.

4. Conclusion

It is shown that in case of small bubbles, the presence of capillary effects and solubility of the gas in the liquid leads to a new resonance frequency that differs from the Minnaert frequency. A simple analytic formula is obtained that relates the resonance frequency of a soluble gas bubble and its radius.

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